Overdetermined elliptic problems and a conjecture of Berestycki, Caffarelli and Nirenberg.

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Joint work with A. Ros and P. Sicbaldi (U. Granada)

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Outline

The problem

We say that a smooth domain $\Omega \subset \mathbb{R}^N$ is extremal if the following problem admits a **bounded solution**:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c < 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\nu(x)$ is the exterior normal vector to $\partial\Omega$ at x, and f is a Lipschitz function.

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If Ω is a bounded extremal domain, then it is a ball and u is radially symmetric.

J. Serrin, 1971.

The BCN Conjecture

The case of unbounded domains was first treated by Berestycki, Caffarelli and Nirenberg in 1997.

They show that the domain must be a half-plane under assumptions of asymptotic flatness of the domain.

In that paper they proposed the following conjecture:

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In that paper they proposed the following conjecture:

- If Ω is a extremal domain and $\mathbb{R}^n \backslash \overline{\Omega}$ is connected, then Ω is either a ball B^n, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{n-k}$, or the complement of one of them.

H. Berestycki, L. Caffarelli and L. Nirenberg, 1997.

The BCN conjecture is false for $N \geq 3$!

This conjecture was disproved for $N \geq 3$ by P. Sicbaldi: he builds extremal domains obtained as a periodic perturbation of a cylinder (for $f(t) = \lambda t$).

P. Sicbaldi, 2010.

F. Schlenk and P. Sicbaldi, 2011

This construction works also for $N = 2$, but in this case $\mathbb{R}^2 \setminus \Omega$ is not connected.

Overdetermined problems and CMC surfaces

A formal analogy with constant mean curvature surfaces has been observed:

- Serrin's result is the counterpart of Alexandrov's one on CMC hypersurfaces.
- Sicbaldi example has a natural analogue in the Delaunay CMC surface.

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Other extremal domains have been built for f of Allen-Cahn type ($f(u) = u - u^3$), with

- $\partial\Omega$ close to a dilated embedded minimal surface in $\mathbb{R}^3$ with finite total curvature and nondegenerate.
- $\partial\Omega$ close to a dilated Delaunay surface in $\mathbb{R}^3$.

M. Del Pino, F. Pacard and J. Wei, 2015.

Overdetermined problems and the De Giorgi conjecture

The case of nonlinearities of Allen-Cahn type has been considered in many papers, in relation with the well-known De Giorgi conjecture.

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A extremal domain has been built with boundary close to the Bombieri-De Giorgi-Giusti minimal graph if $N = 9$. In this example, u is monotone.

M. Del Pino, F. Pacard and J. Wei, 2015.

These solutions do not exist if $N \leq 8$.

K. Wang and J. Wei, 2017.

Other cases have been studied recently:

1. The harmonic case $f = 0$: Alt, Caffarelli, Hauswirth, Helein, Pacard, Traizet, Jerison, Savin, Kamburov, De Silva, Liu, Wang, Wei...
2. Overdetermined problems on manifolds: Espinar, Farina, Mazet, Mao, Fall, Sicbaldi...

The BCN conjecture in dimension 2

In case $N = 2$, there are some previous results:

- If u is monotone and ∇u is bounded, then Ω is a half-plane.
  - A. Farina and E. Valdinoci, 2010.
- If Ω is contained in a half-plane and ∇u is bounded, then the BCN conjecture holds.
  - A. Ros and P. Sicbaldi, 2013.
- If $\partial\Omega$ is a graph and f is of Allen-Cahn type, then Ω is a half-plane.
  - K. Wang and J. Wei, preprint.
- If u is a stable solution (in a certain sense), then Ω is a half-plane.
  - K. Wang, preprint.

A rigidity result in dimension 2

Theorem

If $N = 2$ and $\partial\Omega$ is connected and **unbounded**, *then Ω is a half-plane.*

A. Ros, D.R and P. Sicbaldi, 2017.

Exterior domains

The only remaining case in dimension 2 is that of exterior domains.

Under some restrictions on f and/or u, a exterior extremal domain must be the exterior of a ball:

- A. Aftalion and J. Busca, 1998.
- W. Reichel, 1997.
- B. Sirakov, 2001.

For instance, the conjecture is true for exterior domains if $f(u) = u - u^3$, or if $f = 0$.

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All those results are based on the moving plane technique from infinity. Hence the solution is radially symmetric and monotone along the radius.

Exterior domains

Our initial observation is: there are **radial** solutions which are **not monotone**! Indeed, for any $p > 1$, the Nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + u - u^p = 0, \ u > 0 & \text{in } B_R^c, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{2}$$

admits nonmonotone radial solutions for any $R > 0$.

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admits nonmonotone radial solutions for any $R > 0$.

We will use these solutions to build a counterexample to the BCN conjecture by a local bifurcation argument.

A counterexample in exterior domains

Theorem

Let $N \in \mathbb{N}$, $N \geq 2$, $p \in (1, \frac{N+2}{N-2})$. Then there exist bounded domains $\mathcal{D}$ different from a ball such that the overdetermined problem:

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admits a bounded solution.

- In particular, we answer negatively to the BCN conjecture for $N = 2$.
- The hypothesis "$\partial\Omega$ unbounded" is essential in our previous work.
- Those solutions are unstable.

We need symmetry!

We denote by $\mu_i = i(i+N-2)$ the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$, and $\tilde{\mu}_i$ the subset of eigenvalues for G-symmetric eigenfunctions.

We choose a symmetry group $G \subset O(N)$, so that:

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Some examples:

- If $G = O(m) \times O(N-m)$, $\tilde{\mu}_1 = \mu_2$ and $\tilde{m}_1 = 1$.
- If $N = 2$ and G is the dihedral group $\mathbb{D}_k$, then $\tilde{\mu}_1 = \mu_k$ and $\tilde{m}_1 = 1$.
- If $N = 3$ we can take G as the group of isometries of:

  the tetrahedron ($\tilde{\mu}_1 = \mu_3$ and $\tilde{m}_1 = 1$),
  the octahedron ($\tilde{\mu}_1 = \mu_4$ and $\tilde{m}_1 = 1$),
  the icosahedron ($\tilde{\mu}_1 = \mu_6$ and $\tilde{m}_1 = 1$).

O. Laporte, 1948.

Known facts about the Dirichlet problem

Denote by B_R the ball of radius R. Then, the problem

$$\begin{cases} -\Delta u + u - u^p = 0, \; u > 0 & \text{in } B_R^c, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{4}$$

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Moreover, u_R is nondegenerate and has Morse index 1 in the radial setting. In other words, the eigenvalue problem

$$\begin{cases} -\Delta \phi + \phi - p u_R^{p-1} \phi = \tau \phi & \text{in } B_R^c, \\ \phi = 0 & \text{on } \partial B_R. \end{cases} \tag{5}$$

has no 0 eigenvalue and just one negative one in $H^1_{0,r}(B_R^c)$.

We denote $z_R \in H^1_{0,r}(B_R^c)$ the eigenfunction with negative eigenvalue.

P. Felmer, S. Martínez and K. Tanaka, 2008.

M. Tang, 2003.

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The answer is no. Indeed, one can show that

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Lemma

The Dirichlet problem is nondegenerate in $H^1_{0,G}(B^c_R)$ for small R.

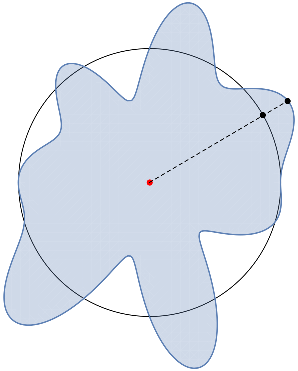
The proof of this Lemma is postponed.

Then the Dirichlet problem is nondegenerate for $R \in (0, R_0)$, where R_0 is the maximal value for that.

The nonlinear Dirichlet-to-Neumann operator

Fix $R \in (0, R_0)$. Given a function $w : \mathbb{S}^{N-1} \longmapsto (0, \infty)$, let us denote B_w its radial graph,

$$B_w := \left\{ x \in \mathbb{R}^N \qquad |x| < w(x/|x|) \right\} .$$



By the Inverse Function Theorem, for all $v \in C^{2,\alpha}_G(\mathbb{S}^{N-1})$ small, there exists a positive solution $u = u(R, v)$ to the problem

$$\begin{cases} -\Delta u + u - u^p = 0 & \text{in} \quad B^c_{R+v} \\ u = 0 & \text{on} \quad \partial B_{R+v} \,. \end{cases}$$

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We define the Dirichlet-to-Neumann operator:

$$F(R, v) = \frac{\partial u}{\partial \nu} - \frac{1}{|\partial B_{R+v}|} \int_{\partial B_{R+v}} \frac{\partial u}{\partial \nu} \, dx,$$

Clearly, we are done if we prove the existence of nontrivial solutions of the equation $F(R, v) = 0$. From now on, we assume that v has 0 mean.

A necessary condition for bifurcation is that $D_v F(R, 0)$ becomes degenerate.

Degeneracy of the linearized operator

$D_vF(R,0)$ is degenerate at a point $(R,0)$ if there exists $\psi \neq 0$ such that:

$$\begin{cases} -\Delta\psi + \psi - pu_R^{p-1}\psi = 0 & \text{in } B_R^c, \\ \frac{\partial \psi}{\partial \nu}(x) - \frac{N-1}{R}\psi(x) = 0 & \text{on } \partial B_R, \end{cases} \tag{6}$$

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Multiplying by z and integrating by parts,

$$\int_{B_R^c} \psi z_R = 0.$$

The quadratic form

The associated quadratic form is $Q = Q_R : E \to \mathbb{R}$,

$$Q(\psi) = \int_{B_R^c} \left(|\nabla \psi|^2 + \psi^2 - p u_R^{p-1} \psi^2 \right) - \frac{N-1}{R} \int_{\partial B_R} \psi^2,$$

$$E = \left\{ \psi \in H_G^1(B_R^c),\ \int_{\partial B_R} \psi = 0,\ \int_{B_R^c} \psi z_R = 0 \right\}.$$

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Let us denote $Q_0 = Q|_{E_0}$ the quadratic form of the Dirichlet problem,

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Proposition

If R is sufficiently small, then Q is positive definite in E.

This result gives us a spectral gap where there is no bifurcation.
Moreover, it shows that the Dirichlet problem is nondegenerate for small R.

Sketch of the proof

The proof is by contradiction; take $R = R_n \to 0$, $B_n = B_{R_n}$, $u_n = u_{R_n}$ and $z_n = z_{R_n}$.

We first prove that $u_n \to U$ and $z_n \to Z$ in H^1 sense, where U is the groundstate of:

$$-\Delta U + U = U^p \quad \text{in } \mathbb{R}^N,$$

and Z is the positive radial solution of

$$-\Delta Z + Z - pU^{p-1}Z = \tau Z \quad \text{in } \mathbb{R}^N,$$

with $\tau < 0$. This is the only point where the assumption $p < \frac{N+2}{N-2}$ is required.

Assume by contradiction that there exist normalized solutions $\psi_n \in E$ of:

$$\begin{cases} -\Delta\psi_n + \psi_n - pu_n^{p-1}\psi = \chi_n\psi & \text{in } B_n^c, \\ \frac{\partial\psi_n}{\partial\eta} - \frac{N-1}{R_n}\psi_n = 0 & \text{on } \partial B_n, \end{cases}$$

with $\chi_n \leq 0$.

Hence there exists $\psi_0 \in H^1(\mathbb{R}^N)$ such that $\psi_n \rightharpoonup \psi_0$ in $H^1(B_r^c)$, for any $r > 0$.

$$\psi_0 \neq 0?$$

Recall the expression of the quadratic form:

$$Q(\psi) = \int_{B_R^c} \left(|\nabla \psi|^2 + \psi^2 - p u_R^{p-1} \psi^2 \right) - \frac{N-1}{R} \int_{\partial B_R} \psi^2,$$

We need to control the boundary term with the Dirichlet energy:

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Lemma

The following inequality holds:

$$\frac{1}{R} \int_{\partial B_R} \psi^2 \leq \frac{1}{N} \int_{B_R^c} |\nabla \psi|^2,$$

for any $\psi \in H_G^1(B_R^c)$ *with* $\int_{\partial B_R} \psi = 0$*.*

Here the G-symmetry is needed!

In the limit, $\psi_0 \neq 0$ is a solution of:

$$-\Delta\psi_0 + \psi_0 - pU^{p-1}\psi_0 = \chi_0\psi_0 \text{ in } \mathbb{R}^N \setminus \{0\},$$

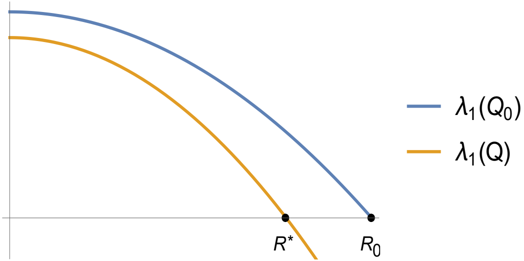
with

$$\int_{\mathbb{R}^N} \psi_0 Z = 0, \; \chi_0 \leq 0.$$

But the singularity is removable, and this is impossible by the known properties of U.

Q becomes degenerate for some R^*

Recall that the Dirichlet problem is nondegenerate for $R \in (0, R_0)$ and Q_0 is positive semidefinite for $R = R_0$.



Therefore the linearized operator becomes degenerate at some $R^* \in (0, R_0)$!

Odd multiplicity

By making Fourier decomposition, we write $\psi = \phi_0(r) + \sum_{i=1}^{+\infty} \phi_i(r)\zeta_i(\theta)$, with $r = |x|$, $\theta = \frac{x}{|x|}$ and ζ_i are $G-$symmetric spherical harmonics.Then,

$$\phi_0(R) = 0, \int_R^{+\infty} \phi_0(r) z_R(r) r^{N-1}\, dr = 0,$$

$$Q(\psi) = \sum_{i=0}^{+\infty} \tilde{Q}_i(\phi_i),$$

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with

$$\tilde{Q}_0(\phi) = \int_R^{+\infty} (\phi'(r)^2 + \phi(r)^2 - p u_R(r)^{p-1}\phi(r)^2) r^{N-1}\, dr - (N-1)R^{N-2}\phi(R)^2,$$

$$\tilde{Q}_i(\phi_i) = \tilde{Q}_0(\phi_i) + \tilde{\mu}_i \int_R^{+\infty} \phi_i(r)^2 r^{N-3}.$$

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2. $\tilde{Q}_1$ is degenerate for $R = R^*$, with 1-D kernel.
3. $\tilde{Q}_i$ are positive definite, $i > 1$.
4. Hence Q is degenerate with kernel of dimension $\tilde{m}_1$ (odd by assumption).

This allows us to use the local bifurcation theorem of Krasnoselskii.

Thank you for your attention!